Millimeter Laser Ranging to the Moon: a comprehensive theoretical model for advanced data analysis

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Next 25 slides:

• The basics of LLR
• Historical background
• The Newtonian Motion
• General Relativity at a glimpse
• PPN equations of motion
• Motivations behind PPN
• Gauge freedom in the lunar motion
• IAU 2000 theory of reference frames
• Lunar theory in a local-inertial reference frame
• Magnitude of synodic relativistic terms
No celestial body has required as much labor for the study of its motion as the Moon!

M – the mean anomaly of the Moon
M' – the mean anomaly of the Sun
D = M - M'

True longitude of the Moon = the mean longitude
(20905 km) days
+ 377' sin M \_ECCENTRIC-1\_ (period 27.3 days)
+ 13' sin 2M \_ECCENTRIC-2\_ (period 13.7 days)
+ 76' sin (2D - M) \_EVECTION\_ (period 31.8 days)
+ 39' sin 2D \_VARIATION\_ (period 14.7 days)

Earth
Moon
Sun

\[ M \] – the mean anomaly of the Moon
\[ M' \] – the mean anomaly of the Sun
\[ D = M - M' \]
Historical Background (before Einstein)

- **Newton** – the first theoretical explanation of the main lunar inequalities (1687)
- **Clairaut** – lunar theory with the precision of 1.5 arc-minute (1752)
- **Laplace** – the lunar theory with the precision of 0.5 arc-minute; secular acceleration; speed of gravity (1802)
- **Hansen** – the lunar theory and tables with the precision of 1 arc-second (1857)
- **Delaune** – an elliptic unperturbed orbit; 230 terms in the perturbing function; perturbation of the canonical set of elements; precision 1 arc-second (1860)
- **Hill** – rotating coordinates; Hill’s equation; Hill’s intermediate orbit; precision 0.1 arc-second (1878)
- **Brown** – extension of Hill’s theory; Brown’s tables; precision 0.01 arc-second (1919)
Historical Background (after Einstein)

- **De Sitter** – relativistic equations of the Moon; geodetic precession (1919)
- **Einstein-Infeld-Hoffmann** – relativistic equations of N-body problem; massive bodies as singularities of space-time (1938)
- **Fock-Petrova** – relativistic equations of N-body problem; massive bodies as extended fluid balls (1940)
- **Brumberg** – relativistic Hill-Brown theory of the Moon based on the EIH equations; eccentricity in relativistic term $e = 0$ (1958)
- **Baierlein** – extension of Brumberg’s theory for $e \neq 0$ (1967)
- **Apollo 11** - LLR technique gets operational; ranging precision = a few meters (1969)
- **Nordtvedt** – testing the strong principle of equivalence with LLR (1972)
- **Standish** – JPL numerical ephemerides of the Moon and planets (DE/LE)
- **IAU 2000** – relativistic resolutions on time scales and reference frames based on the BK-DSX papers
- **APOLLO** – new LLR technology at the Apache Point Observatory (2005); ranging precision 1 millimeter
Newtonian Equations of the Lunar Motion

\[
\frac{d^2 \vec{x}_L}{dt^2} = \vec{\nabla} U_E(\vec{x}_L) + \vec{Q}_L + \vec{\nabla} U(\vec{x}_L)
\]

Earth's gravity  Figure's effects  Sun and planets

\[
\frac{d^2 \vec{x}_E}{dt^2} = \vec{\nabla} U_L(\vec{x}_E) + \vec{Q}_E + \vec{\nabla} U(\vec{x}_E)
\]

Moon's gravity  Figure's effects  Sun and planets

\[
M = M_L + M_E; \quad M\vec{x}_{cm} = M_L \vec{x}_L + M_E \vec{x}_E; \quad \vec{r} = \vec{x}_L - \vec{x}_E;
\]

\[
\frac{d^2 \vec{x}_{cm}}{dt^2} = \frac{M_L}{M} \vec{\nabla} U(\vec{x}_L) + \frac{M_E}{M} \vec{\nabla} U(\vec{x}_E) = \vec{\nabla} U(\vec{x}_{cm}) + \text{(tidal terms)}
\]

\[
\frac{d^2 \vec{r}}{dt^2} = \vec{\nabla} \left[U_E(\vec{x}_L) - U_L(\vec{x}_E)\right] + \vec{\nabla} \left[U(\vec{x}_L) - U(\vec{x}_E)\right]
\]

Earth-Moon gravity force  Tidal gravity force from the Sun and planets. Gradient of the perturbing potential.
Gravitational Field is not a Scalar!
From Minkowski to Riemann geometry

\[ ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]

\[ ds^2 = g_{00} c^2 dt^2 + 2 g_{0i} c dt dx^i + g_{ij} dx^i dx^j \]
General Theory of Relativity at a glimpse

• The metric tensor $\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left( \frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right)$

• The affine connection $\nabla$ the force of gravity

• The Riemann tensor $\equiv$ the relative (tidal) force of gravity

• The Principle of Equivalence $\nabla$ the covariant derivative $\nabla$

• The Gravity Field Equations

\[ R^\alpha_{\beta} - \frac{1}{2} \delta^\alpha_{\beta} R = \frac{8\pi G}{c^4} T^\alpha_{\beta} \]

\[ \nabla_\alpha G^\alpha_{\beta} \equiv 0 \Rightarrow \nabla_\alpha T^\alpha_{\beta} = 0 \]

Matter tells space-time how to curve: field eqs.

Space-time tells matter how to move: eqs. of motion
PPN metric tensor for a spherical body

\[
\begin{align*}
g_{00} &= -1 + 2\alpha \frac{U}{c^2} - 2\beta \frac{U^2}{c^4} \\
g_i := g_{0i} &= 4\mu \frac{(J \times r)_i}{c^3 r^3} \\
g_{ij} &= (1 + 2\gamma) \frac{U}{c^2}
\end{align*}
\]

PPN parameters

Conventional tests of the metric tensor

<table>
<thead>
<tr>
<th>Test</th>
<th>Experiment</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravitational redshift</td>
<td>GP-A</td>
<td>[</td>
</tr>
<tr>
<td>Perihelion shift</td>
<td>Astrophys. observation</td>
<td>[\left</td>
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<tr>
<td>Light deflection</td>
<td>VLBI</td>
<td>[</td>
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<td>Gravitational time delay</td>
<td>Cassini</td>
<td>[</td>
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<tr>
<td>Lense–Thirring</td>
<td>LAGEOS</td>
<td>[</td>
</tr>
<tr>
<td>Schiff</td>
<td>GP-B</td>
<td>[</td>
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</tbody>
</table>
EIH equations of motion

\[ \ddot{a}_i = \ddot{g}_i \]

the Newtonian gravity force

\[- \sum_{j \neq i} \left( \sum_{k \neq i} \frac{\mu_k}{r_{ik}} + \sum_{k \neq j} \frac{\mu_k}{r_{jk}} \right) \ddot{g}_{ij} + 4 \sum_{j \neq i} \dot{v}_i \times \left( \dot{v}_j \times \ddot{g}_{ij} \right) + \]

non-linearity of the gravity field "gravitomagnetic-like" force

\[ + \frac{1}{2} \sum_{j \neq i} \left[ 3v_i^2 + 4v_j^2 - 3 \left( \dot{v}_i \cdot \dot{r}_{ij} \right)^2 \ddot{g}_{ij} - 6 \left( \ddot{g}_{ij} \cdot \dot{v}_j \right) \left( \ddot{v}_j - \dot{v}_i \right) + \left( \ddot{g}_{ij} \cdot \dot{v}_i \right) \dot{v}_i \right] \]

special-relativistic corrections to the gravity force

\[ + \frac{1}{2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}} \left[ 7 \ddot{a}_j + \left( \ddot{a}_j \cdot \dot{r}_{ij} \right) \dot{r}_{ij} \right] - \frac{1}{2} v_i^2 \ddot{a}_i + \left( \ddot{a}_i \cdot \dot{v}_i \right) \dot{v}_i + 3 \sum_{j \neq i} \frac{\mu_j}{r_{ij}} \ddot{a}_i \]

an inductive acceleration-dependent gravity force

the post-Newtonian modification of E\( = m c^2 \)
**PPN equations of motion of extended bodies**

\[
\ddot{a}_i = \left[1 + \frac{\dot{G}}{G}(t-t_0)\right] \frac{(1-\eta \Pi_i)}{\text{violation of SEP}} \ddot{g}_i + \frac{\dot{\eta}}{G} \ddot{g}_i
\]

- Time-dependent $G$
- Newtonian gravity force

\[-(2\beta - 1) \sum_{j \neq i} \left(\sum_k \frac{\mu_k}{r_{ik}} + \sum_{k \neq j} \frac{\mu_k}{r_{jk}}\right) \ddot{g}_{ij} + (2\gamma + 2 - \eta_G) \sum_{j \neq i} \ddot{v}_i \times (\ddot{v}_j \times \ddot{g}_{ij}) + \ldots\]

- "Gravitomagnetic-field" parameter introduced by Soffel et al. (PRD 2008)
- Non-linearity of the gravity field
- "Gravitomagnetic-like" force

\[\frac{1}{2} \sum_{j \neq i} \left[ (2\gamma + 1)\dot{v}_i^2 + (2\gamma + 2)\dot{v}_j^2 - 3 (\ddot{v}_i \cdot \dddot{r}_{ij})^2 \ddot{g}_{ij} - (4\gamma + 2) (\ddot{g}_{ij} \cdot \dddot{v}_j) (\dddot{v}_j - \dddot{v}_i) + (\ddot{g}_{ij} \cdot \dddot{v}_i) \ddot{v}_i \right] \]

- Lorentz-invariance of the gravity force (preferred frame effects)

\[\frac{1}{2} \sum_{j \neq i} \frac{\mu_j}{r_{ij}} \left[ (4\gamma + 3) \dddot{a}_j + (\dddot{a}_j \cdot \dddot{r}_{ij}) \dddot{r}_{ij} \right] - \frac{1}{2} \dot{v}_i^2 \dddot{a}_i + (\dddot{a}_i \cdot \dddot{v}_i) \dddot{v}_i + (2\gamma + 1) \sum_{j \neq i} \frac{\mu_j}{r_{ij}} \dddot{a}_i\]

- An inductive acceleration-dependent gravity force
- The post-Newtonian modification of $E = mc^2$

Solution of these equations must be substituted to the solution of equation of a laser pulse propagation (time-delay equation). The PPN time-delay equation has many terms being identical to those in the PPN equations of motion of extended bodies.
‘Conventional’ PPN ranging model

• Any coordinate reference system can be used in relativity to interpret the data.
  True, but making use of inappropriate coordinates easily leads to misinterpretation of gravitational physics.

• Modern computer technology is highly advanced. Data processing can be done in any coordinates irrespectively of the complexity of the equations of motion.
  True, but making use of inappropriate coordinates mixes up the spurious, gauge-dependent effects with real physical effects and makes them entangled. There is no unambiguous way to clearly separate gravitational physics from coordinate effects.

• Any post-Newtonian term in the PPN equations of motion has physical meaning and, in principle, can be measured.
  Not true. The PPN equations of motion of the Moon have an enormous number of spurious, gauge-dependent terms that have no physical meaning.
The gauge condition is imposed on the metric tensor. It simplifies the gravity field equations making their solution mathematically simpler. However, the residual gauge freedom remains. It is defined by the gauge functions $\xi^\alpha$, which obey certain equations and introduce a number of spurious (unphysical) terms to the metric tensor (= gravity field potentials)

\[
\overbrace{w^\alpha}^{\text{'new' coordinates}} = \overbrace{x^\alpha}^{\text{'old' coordinates}} + \overbrace{\xi^\alpha(x)}^{\text{the gauge functions}}
\]

\[
g_{\alpha\beta}(x) = g_{\mu\nu}(w) \frac{\partial w^\mu}{\partial x^\alpha} \frac{\partial w^\nu}{\partial x^\beta} = g_{\alpha\beta}(w) + \xi_{\alpha,\beta} + \xi_{\alpha,\beta} + O(\xi^2)
\]

The spurious terms enter relativistic equations of motion of both the bodies and photons. They must be carefully disentangled from the real physical effects existing in the motion of the celestial bodies. The Moon-Earth-Sun system admits a large number of the gauge degrees of freedom, which can be eliminated after transformation to the local inertial frame of the EM barycenter.
Lorentz and Einstein contractions as the gauge modes

Magnitude of the contractions is about 1 meter!
Ellipticity of the Earth’s orbit leads to their annual oscillation of about 2 millimeters. Are they observable by means of LLR?
Shape of a moving body can be defined in the global frame but it faces major difficulties because of the Lorentz contraction and other (non-linear) frame-dependent coordinate effects. One needs a local frame to work out a such definition.

To maintain the shape of the celestial body in the global frame, one has to introduce a spurious stress and strain inside the body to compensate the Lorentz contraction (physics does not work in this way).
Ranging model of a gauge-invariant theory of gravity

Earth  \( \vec{r}(t_1) \)  Moon

\( \vec{x}_E(t_1) \)
\( \vec{x}_L(t_2) \)

Sun

Solar system barycenter

\[
\begin{align*}
\vec{x}_L(t_2) - \vec{x}_E(t_1) &= \text{Newtonian orbit} + \text{Gauge-dependent terms} + \text{Physical PN perturbations} \\
\vec{r}(t_1) &= \text{Newtonian ERP} + \text{Gauge-dependent terms} + \text{Physical PN perturbations} \\
\vec{p}(t_2) &= \text{Newtonian LRP} + \text{Gauge-dependent terms} + \text{Physical PN perturbations} \\
c(\tau_2 - \tau_1) &= (\vec{x}_L(t_2) - \vec{x}_E(t_1) + \vec{p}(t_2) - \vec{r}(t_1)) + \text{PN time delay (Sun)} + \text{PN time delay (Earth)}
\end{align*}
\]

Gauge-independent observable time delay

contains the gauge-dependent terms

contains the gauge-dependent terms

all together these terms are gauge-independent

What is happening in the ‘conventional’ PPN ranging model?

Earth \[ \vec{r}(t_1) \] to Moon \[ \vec{\rho}(t_2) \] via Sun \[ \vec{x}_E(t_1), \vec{x}_L(t_2) \] with the range:

\[
\vec{r}(t_1) = \text{Newtonian ERP} + \text{Gauge-dependent terms} + \text{Physical PN perturbations}
\]

\[
\vec{\rho}(t_2) = \text{Newtonian LRP} + \text{Gauge-dependent terms} + \text{Physical PN perturbations}
\]

\[
c(\tau_2 - \tau_1) = |\vec{x}_L(t_2) - \vec{x}_E(t_1) + \vec{\rho}(t_2) - \vec{r}(t_1)| + \text{PN time delay (Sun)} + \text{PN time delay (Earth)}
\]

\[
\text{Gauge-independent observable time delay}
\]

\[
\text{contains the gauge-dependent terms}
\]

\[
\text{contains the gauge-dependent terms}
\]

all together these terms are NOT gauge-independent but proportional to \((\eta_G - 1)\)
Correcting the PPN ranging model

\[ \bar{x}_L(t_2) - \bar{x}_E(t_1) = \text{Newtonian orbit} + \eta_G \text{Gauge-dependent terms} + \text{Physical PN perturbations} \]

\[ \bar{r}(t_1) = \text{Newtonian ERP} + \eta_G \text{Gauge-dependent terms} + \text{Physical PN perturbations} \]

\[ \bar{\rho}(t_2) = \text{Newtonian LRP} + \eta_G \text{Gauge-dependent terms} + \text{Physical PN perturbations} \]

\[ c(\tau_2 - \tau_1) = |\bar{x}_L(t_2) - \bar{x}_E(t_1) + \bar{\rho}(t_2) - \bar{r}(t_1)| + \eta_G \]

PN time delay (Sun) + PN time delay (Earth)

Some details in: Kopeikin & Vlasov, Physics Reports, 2004
<table>
<thead>
<tr>
<th>Term</th>
<th>Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schwarschild</td>
<td>( \frac{GM_\oplus}{c^2} ) \quad : 1 \text{ cm}</td>
</tr>
<tr>
<td>Lense-Thirring</td>
<td>( \frac{\omega_\oplus R_\oplus v}{c} \frac{R_\oplus}{c} ) \quad 0.3 \text{ mm}</td>
</tr>
<tr>
<td>PN Quadrupole</td>
<td>( \frac{GM_\oplus}{c^2} \left( \frac{R_\oplus}{r} \right)^2 J_{2\oplus} ) \quad 2 \times 10^{-4} \text{ mm}</td>
</tr>
<tr>
<td>Gauge-dependent terms</td>
<td>( \frac{v}{c} \frac{v_\oplus}{c} r + \ldots ) \quad \text{from a few meters down to a few mm}</td>
</tr>
<tr>
<td>PN Gravitomagnetic</td>
<td>( \left( \frac{n_\oplus}{n_\oplus} \right)^2 \frac{v}{c} \frac{v_\oplus}{c} r ) \quad a few mm</td>
</tr>
<tr>
<td>PN Gravitoelectric</td>
<td>( \left( \frac{n_\oplus}{n_\oplus} \right)^2 \left( \frac{v_\oplus}{c} \right)^2 r ) \quad a few cm</td>
</tr>
<tr>
<td>Non-linearity of gravity</td>
<td>( \left( \frac{n_\oplus}{n_\oplus} \right)^2 \frac{GM_\oplus}{c^2} ) \quad 0.1 \text{ mm}</td>
</tr>
</tbody>
</table>
Field equations for the metric tensor

PN approximation

Gauge and boundary conditions

Global frame (BCRF) (t, x)
Resolution B1.3

Coordinate transformation (t, x) \rightarrow (u, w)
Resolutions B1.3 and B1.5

Matching metric tensor in two frames. Residual gauge freedom

Laws of conservation

Translational and rotational equations of motion

Local frame (GCRF) (u, w)
Resolution B1.3

Multipole moments Resolution B1.4
The gravitomagnetic influence on Earth-orbiting spacecrafts and on the lunar orbit

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Gravitomagnetic field is covariantly split in the intrinsic and extrinsic parts, which are generated by rotational and translational currents of matter respectively. The intrinsic component has been recently discovered in the LAGEOS spacecraft experiment. We discuss the method of detection of the extrinsic tidal component with the lunar laser ranging (LLR) technique. Analysis of the gauge residual freedom in the relativistic theory of three-body problem demonstrates that LLR is currently not capable to detect the extrinsic gravitomagnetic effects which are at the ranging level of few millimeters. Its detection requires further advances in the LLR technique that are coming in the next 5-10 years.

PACS numbers: 04.20.-q, 04.80.Cc, 96.25.De

Submitted to the book in memory of J.A. Wheeler:
Editor: I. Ciufolini (2009)
Reference Frames

Global RF $(t, x^i)$

Local RF $(u, w^i)$

Observer RF $(\tau, \xi^i)$

World-line of the geocenter

Geodesic world-line

Earth

Moon

Sun

Jupiter

CCR RF $(\hat{\tau}, \hat{\xi}^i)$

\[ \tau = \tau(u, w) \]
\[ u = u(t, x) \]
\[ \xi^i = \xi^i(u, w) \]
\[ w^i = w^i(t, x) \]
Lunar theory in the local-inertial frame.

- Earth-Moon system being considered locally, is a binary system on a curved space-time background (Sun, planets).
- Equations of motion of the Earth-Moon system are those of the deviation of geodesics perturbed by the mutual gravitational interaction between Earth and Moon.
- There is a considerable similarity between this problem and that of the evolution of the cosmological perturbations in expanding universe.
- Earth-Moon equations of motion have enormous gauge freedom leading to spurious gauge-dependent modes in motion of the celestial bodies participating in three-body problem.
- The main goal of the advanced lunar theory is
  - to remove all gauge modes,
  - to construct and to match reference frames in the Earth-Moon system with a sub-millimeter tolerance,
  - to ensure that ‘observed’ geophysical parameters and processes are real.
- This is not trivial mathematical problem that requires a peer attention of experts in relativity!
Relativistic mass, center-of-mass and the Earth/Moon figure

- Definition of mass, center of mass and other multipoles must include the post-Newtonian corrections
- Definition of the body’s local reference frame
- Definition of figure in terms of distribution of intrinsic quantities: density, energy, stresses
- Relativistic definition of the equipotential surface – geoid/celenoid (Kopeikin S., 1991, Manuscripta Geodetica, 16, 301)
Rotation of the Earth/Moon in the Local Frame
(Kopeikin & Vlasov, Physics Reports, 2004)

- Define the intrinsic angular momentum $\mathbf{S} = \mathbf{I} \cdot \mathbf{\Omega}$ of the rotating body in the locally-inertial frame of the body
- Derive equations of the rotational motion in the locally-inertial frame of the body

\[
\frac{d\mathbf{S}}{d\tau} = (\text{body's quadrupole}) \times (\text{tidal octupole of the Sun, Earth and planets}) + \ldots + \frac{1}{c^2} (\text{post-Newtonian relativistic torque}) + \frac{1}{c^4} (\text{neglectibly small})
\]
This is the last slide.

THANK YOU!