

IS YOUR PERFORMANCE BEING RUINED BY INTERPOLATION ERRORS?

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INTRODUCTION

Improper interpolation methods can have devastating effects on Normal Point (NP) accuracy, telescope pointing accuracy, and indeed any other performance measure.

The issue was raised by Werner Gurtner at the 13th Workshop (Gurtner, 2002), was subsequently discussed by the ILRS Prediction Format Study Group on 9 April 2003 in Nice, France (*Seemueller 2003*), and presented at the Koetzing Workshop (*Luck 2003*). This study quantifies the effects of interpolation errors on the accuracy of SLR NPs, and also looks at LLR and at telescope pointing errors.

The crux of the issue is the tabular interval used for interpolating the predictions used in generating the NPs. The type of interpolating function (cubic spline or Lagrange) and its degree are also important. For example, the Orroal predictions were integrated in steps equal to the NP bin sizes recommended by ILRS, which thus formed the tabular points on which subsequent interpolations were based. The tabular intervals between these points are much too large for successful low-order interpolation, and can produce many **nanoseconds** of error in the shot-by-shot predictions and **minutes of arc** in the pointing. It is shown that the effects on NP generation can amount to errors of several centimeters, in the worst case being 1/3 of the interpolation error.

It is recommended to tighten the ILRS Normal Point Algorithm. Several addenda address interpolation errors in pointing angles which can be large and oscillatory; the best order of Lagrange interpolation to use, and a computationally convenient version of Lagrange's formula.

INTERPOLATION ERRORS

Test data

The satellite data used in this study were kindly supplied by Chris Moore of EOSSS. They consist of the predictions in range, azimuth and elevation integrated from real IRVs in steps of 1 second. These 1-second points are taken to be "truth", i.e. perfect. The tabular points for the interpolation tests were then simply chosen as every k -th point from that set, where the tabular interval is k seconds. They are equally spaced.

The tests consisted of interpolating using cubic splines, which have desirable continuity properties, and Lagrange's formula based on 4 points (order 4), i.e. a cubic to match the splines, and on 6 (order 6) or more points, then subtracting the "true" values from the outcomes. See for example (*Dahlquist & Bjorck 1974*).

For LAGEOS predictions, tabular intervals of 60 seconds were used rather than the ILRS bin size of 120 seconds. For Apollo-15, either the old value of 900 seconds, or 120 seconds, was used, generated by a revision of the classical program EULER.

Test results

The results are summarized in Table 1. The following graphs are a selection of those generated. They cover a variety of maximum elevations. The maximum interpolation errors are often several nanoseconds, or at best several hundred picoseconds. The improvements caused by increasing the Lagrange order (Fig. 1), and by reducing the tabular interval (Fig. 3), are evident by comparing the ‘Range Error’ scales of the graphs. Fig. 2 illustrates the typical periodicity in the error, while Fig. 4 shows how bad a perfectly ordinary pass can be. Fig. 2 reveals that spline interpolation errors are smooth, whereas Fig. 1(a) shows the spikiness of Lagrange interpolation errors at the tabular points. Fig. 5 illustrates that lunar ranging is not exempt from these errors.

Table 1: MAXIMUM INTERPOLATION ERROR MAGNITUDES, Cubic Splines, and Lagrangians of given order

PASS	Max. El (deg)	Num Pts	Tab. Int (sec)	Lagr. order	Az. Error (arcsec)		El. Error (arcsec)		Range Error (nanosec)	
					Spln	Lagr	Spln	Lagr	Spln	Lagr
AJISAI-1	87.7	1468	15	4	Huge	huge	420	340	1.0	9.5
			15	6	Huge	huge	415	410	3.2	0.5
AJISAI-2	72.8	1451	5	10	-	-	-	-	.011	tiny
LGEOS1-1	84.5	4183	60	4	Huge	huge	huge	huge	1.5	11.8
			60	6	600	950	55	75	1.3	0.04
			60	10	-	400	-	160	-	tiny
LGEOS1-2	31.0	3483	60	4	-	-	-	-	2.3	6.0
			60	6	-	-	-	-	2.3	tiny
ETLON1	62.8	2536 3	200	4	-	-	-	-	0.75	1.45
			200	6	-	-	-	-	0.75	tiny
STELLA	37.8	887	15	6	0.5	0.2	0.12	0.08	3.6	0.8
CHAMP-1	25.4	582	10	4	0.3	2.2	0.2	1.6	2.7	24.0
			10	6	0.3	0.1	0.2	0.03	2.7	0.6
			5	6	-	-	-	-	0.15	0.01
CHAMP-2	51.7	615	5	4	0.55	4.5	0.3	2.5	0.7	6.5
			5	6	0.55	0.3	0.3	tiny	0.7	0.1
APO-15	82.1	917	900	4	70.	390	5	20	11.5	2.3
			900	10	-	38	-	2	-	0.09
		5533	120	4	0.02	0.17	0.02	0.02	0.7	0.73
			120	10	0.02	0.08	0.02	0.02	0.7	0.07

Note that Cubic Splines (order 4) outperform Lagrange Interpolation of order 4. However, Lagrange Interpolations of order 6 outperform Cubic Splines substantially, and even higher order Lagranges do vastly better.

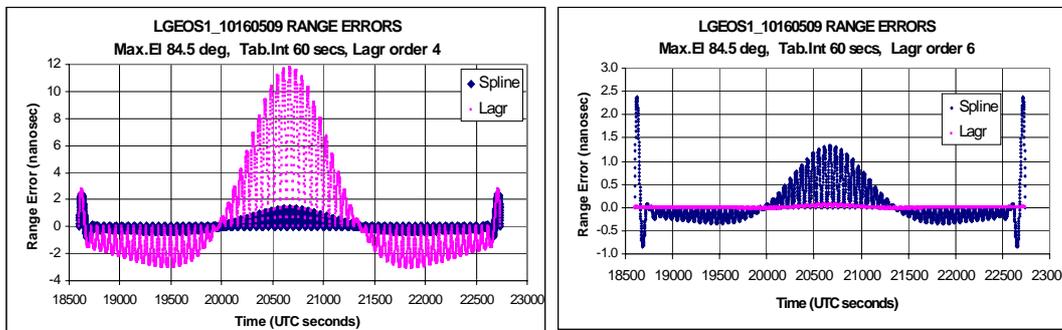


Figure 1: High LAGEOS-1 pass. Note poor behaviour of both at start and end. (a) Lagrange order 4 to match cubic splines. (b) Lagrange order 6.

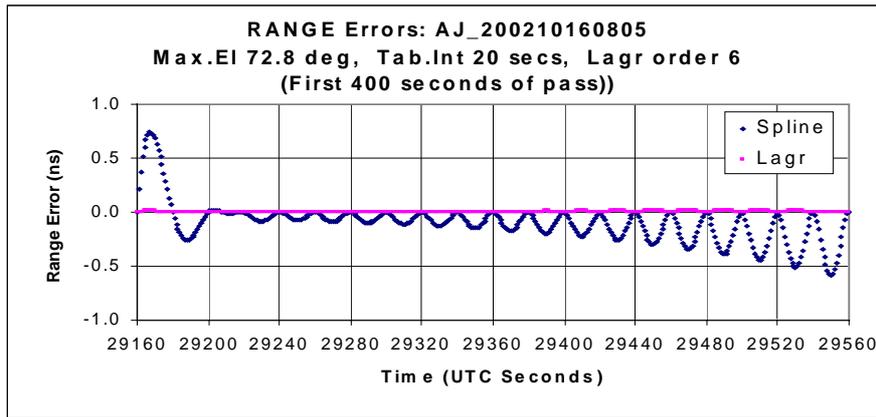


Figure 2: The start of a fairly high AJISAI pass, clearly showing the error periodicity.

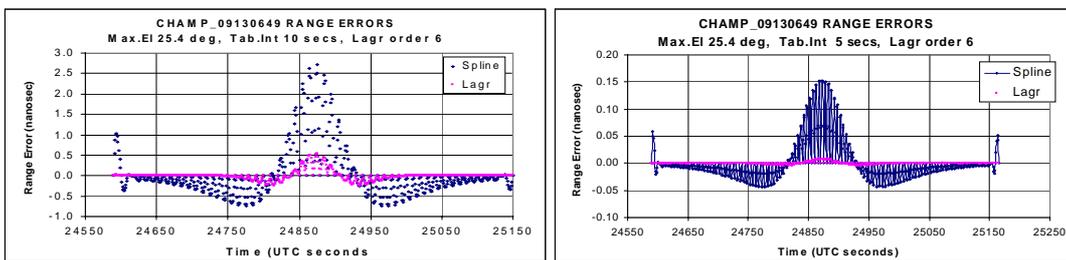


Figure 3: Range interpolation errors for a low CHAMP pass, Lagrange order 6. (a) at 10-second tabular interval. (b) at 5 second tabular interval. Note the 'Range Error' scales!

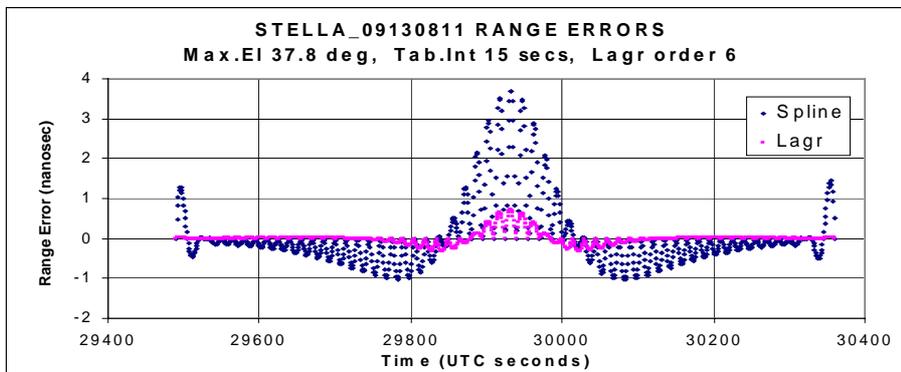


Figure 4: A moderate-elevation STELLA pass. Lagrange errors are significant even at order 6.

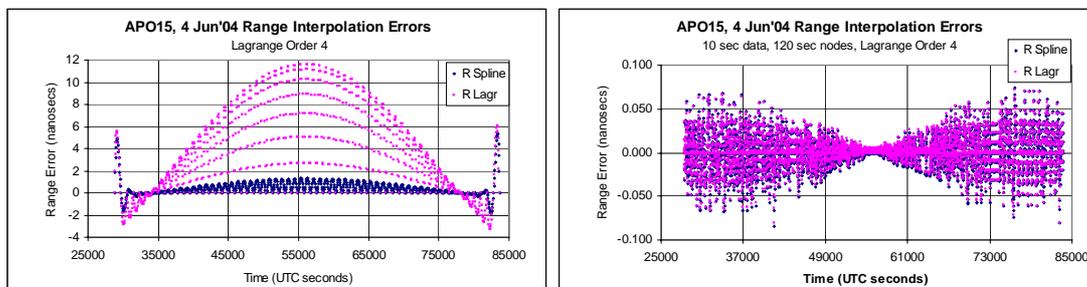


Figure 5: Apollo-15 Range interpolation errors for Lagrange order 4. (a) 900 second, and (b) 120 second tabular intervals.

NORMAL POINT GENERATION

ILRS Algorithm

The 1997 ILRS Normal Point Algorithm is given in the website:

http://ilrs.gsfc.nasa.gov/products_formats_procedures/normal_point/np_algo.html .

It is summarized here and in Fig. 6, with some expansion of notation.

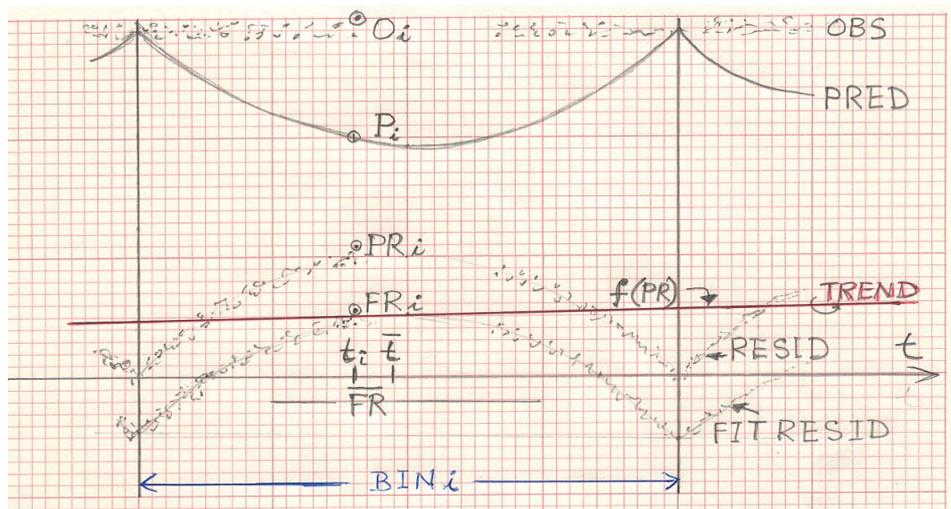


Figure 6: The NP Algorithm in the presence of interpolation error.

- (1) Use high precision predictions to generate **prediction residuals** $PR_j = O_j - P_j$ where j indexes all returns in the pass.
- (2) Use a suitable range window to remove large outliers.
- (3) Solve for a set of parameters . . . to remove the systematic **trends** of the prediction residuals, giving the **trend function** $f(PR)$.
- (4) Compute **fit residuals** $FR_j = PR_j - f(PR_j)$ and omit outliers.
- (5) Iterate steps (3) and (4) until the process converges.
- (6) Subdivide the accepted fit residuals FR_j into bins at fixed intervals.
- (7) Compute the **mean value** \overline{FR}_i , the **mean epoch** \bar{t} and the **number** n_i within a bin i .
- (8) Locate the **particular observation** O_i with its fit residual FR_i and epoch t_i such that $|t_i - \bar{t}|$ is minimum.
- (9) The **NP** is computed as $NP_i = O_i - FR_i + \overline{FR}_i$.
- (10) Compute the RMS_i for bin i .
- (11) Report t_i, NP_i, n_i, RMS_i .

Simple Case

To clarify the argument without (I believe) compromising its general validity, suppose that, in a single bin i , the observed ranges happen to be approximately constant with values:

$$O_j = O_0 + \delta O_j \text{ where } \sum_{j \in \text{bin}(i)} \delta O_j = 0$$

as shown in Fig. 6. In particular, the observed value of the point selected in step (8) above is:

$$O_i = O_0 + \delta O_i,$$

and the value of the NP should be simply O_0 . The prediction residual PR_i for this point is then, from step (1):

$$PR_i = O_0 + \delta O_i - P_i.$$

The trend function is fitted through all the PR_j in the pass after filtering (steps (3) and (4)). It therefore smoothes out the short-period interpolation errors illustrated in Figs. 1-5. Suppose that it happens to be very well-behaved in bin i , in fact constant:

$$f(PR_j) = f.$$

Then the fit residual (step (4)) for the selected observation is:

$$\begin{aligned} FR_i &= PR_i - f \\ &= O_0 + \delta O_i - P_i - f \end{aligned}$$

while the average fit residual within the bin (step (7)) is:

$$\overline{FR} = O_0 - \overline{P} - f \quad \text{since} \quad \sum \delta O_j = 0.$$

Step (9) then yields after some simple manipulation:

$$NP_i = O_0 + (P_i - \overline{P}).$$

The true result is therefore perturbed by the prediction error $(P_i - \overline{P})$.

Magnitude of the Prediction Error

From the graphs, it seems reasonable to model as a first step the interpolation errors by a quadratic function between the tabular points at t_0, t_1 bounding bin i . (It is more likely a quartic, especially for cubic splines.) Let the prediction interpolation error be denoted by ΔP_i so that at the representative point:

$$\begin{aligned} P_i &= P_i(\text{true}) + \Delta P_i, \text{ and similarly} \\ \overline{P} &= \overline{P}(\text{true}) + \overline{\Delta P}, \text{ averaged over } j \in \text{bin}(i) \end{aligned}$$

Since i is selected close to the mean point, we have:

$$\begin{aligned} P_i(\text{true}) &\approx \overline{P}(\text{true}) \\ \therefore P_i - \overline{P} &\approx \Delta P_i - \overline{\Delta P} \end{aligned}$$

The interpolation error model will thus be:

$$\Delta P_j = 4\beta(t - t_0)(t_1 - t)$$

which is zero at $t = t_0$ and $t = t_1$, and its maximum value is:

$$\Delta P_j(\text{max}) = \beta(t_1 - t_0)^2 \quad \text{at } t = (t_0 + t_1)/2.$$

Its mean value is:

$$\begin{aligned} \overline{\Delta P} &= \int_{t_0}^{t_1} \Delta P_j dt / \int_{t_0}^{t_1} dt \text{ if considered uniformly dense} \\ &= \frac{2}{3} \beta(t_1 - t_0)^2 \end{aligned}$$

Hence the Normal Point Error can be as large as:

$$\Delta NP_i(\text{max}) \equiv \Delta P_i(\text{max}) - \overline{\Delta P} = \frac{1}{3} \beta(t_1 - t_0)^2 \quad (1)$$

and β is read from the relevant graphs above. From the examples shown, the errors can be many centimeters.

Because the distribution of observations within bins is usually non-uniform, and because the NP error reduces to zero at the tabular points (net points, nodes, knots), it will behave something like a random variable with unknown properties but no greater than this maximum. The effects on NP accuracy, and on the ILRS metric “Normal Point Precision”, could be disastrous.

SOLUTIONS

There are many possible solutions, based on the well-known properties of the errors in polynomial interpolation. If the formula uses n points (degree $n-1$), the tabular interval is h , the n^{th} derivative at some point ξ in (t_0, t_1) is $f^{(n)}(\xi)$ and the underlying interpolating polynomial is $p_n(t)$, the error $\varepsilon(t)$ is:

$$\varepsilon(t) = h^n p_n(t) f^{(n)}(\xi) / n! \quad (2)$$

The solutions include:

More Appropriate Interpolation Formulae

1. Increase the order n . This can be done easily with Lagrange-based formulae such as Newton, Bessel, Everett, etc., but is not available with cubic splines.
2. Reduce the tabular interval h . In the context of this study, it implies that the step size of the orbit integration must be no greater than h . Since the NP bin sizes are set by ILRS, it also implies that any one NP bin will generally contain several tabular points. It is NOT RECOMMENDED to retain a larger integration step-size and reduce the tabular interval by interpolation! Decreasing the integration step-size does not add significant rounding error (Chris Moore, private communication, 2003).

Different Trend Function Regime

3. We note that step (3) of the ILRS Procedure - iteratively fit a trend function - is designed mainly for filtering, but it is also used for calculating the mean fit residual within a bin in step (7). If a new trend function, e.g. a quadratic, is calculated from only the prediction residuals within each bin, it automatically removes (much of) the interpolation error.

Interpolate on Cartesian Coordinates

4. The 1-second files available for this study only contained range, azimuth and elevation. It was found that, by converting them back to topocentric Cartesian coordinates (East, North, Up) for the interpolations, the errors reduced dramatically. This can be readily understood by noting that the conversion equations:

$$\begin{aligned} range &= \sqrt{X^2 + Y^2 + Z^2} \\ azimuth &= \tan^{-1}(X / Y) \\ elevation &= \sin^{-1}(Z / range) \end{aligned} \quad (3)$$

are not at all well represented by polynomials of any reasonable degree. See Addendum 2.

5. Werner Gurtner and Chris Moore have found that interpolating on Geocentric Cartesian coordinates, then converting to range, azimuth and elevation on every shot, also produces dramatic improvements. This is studied in more detail in Addendum 3.

CONCLUSIONS

The accuracy and precision of NPs may be corrupted badly by inappropriate interpolation of the predictions. The magnitude of the resulting NP error in any bin may be as much as 1/3 of the magnitude of the interpolation error (see equation (1)) and amount to very many centimeters. They behave as unmodeled random errors so may be quite difficult to detect.

It is crucial to use tabular intervals as small as possible, consistent with memory size and speed limitations; and to take great care in choosing an interpolation formula which is appropriate to the function tabular interval being interpolated. The degree of the interpolating polynomial must not be too low nor too high.

Interpolation into tables of range, azimuth and elevations should be avoided. Rather, interpolate on Cartesian X,Y,Z coordinates.

The ILRS NP Procedure requires “high precision predictions” but is otherwise not specific and does not, as written, cope adequately with the problem.

RECOMMENDATIONS

1. It is recommended that this issue be considered by the Data Formats and Procedures Working Group, and that appropriate warnings be included in the ILRS Normal Point Algorithm document.
2. Interpolation should be performed on Cartesian coordinates, NOT on azimuth, elevation and range, using a Lagrange (or equivalent) interpolator of order 8.

ACKNOWLEDGEMENTS

Werner Gurtner’s work alerted me to the potential of problems with algorithms used both in SLR operations and in post-processing, including on-site NP generation and star calibrations for telescope pointing accuracy. This report confirms his findings.

Chris Moore extracted the data needed.

REFERENCES

- Dahlquist, G. and Bjorck, A.: “*Numerical Methods*”, Prentice-Hall, Inc. (1974)
- Gurtner, W.: “*Interpolation of Ephemerides*”, 13th International Workshop on Laser Ranging, Washington D.C., 7-11 October 2002
- Luck, J.McK.: “*Interpolation Errors in Normal Point Generation*”, EUROLAS Workshop, Koetting, October 2003
- Ricklefs, R.L.: “*Consolidated Laser Ranging Prediction Format*”, prepared for the ILRS Prediction Format Study Group of the ILRS Data Formats and Procedures Working Group, San Fernando, June 2004 *et seq.*

ADDENDUM 1: INTERPOLATION ERRORS IN POINTING ANGLES

The same problem affects pointing. Extreme examples are shown in Figs. 7-9.

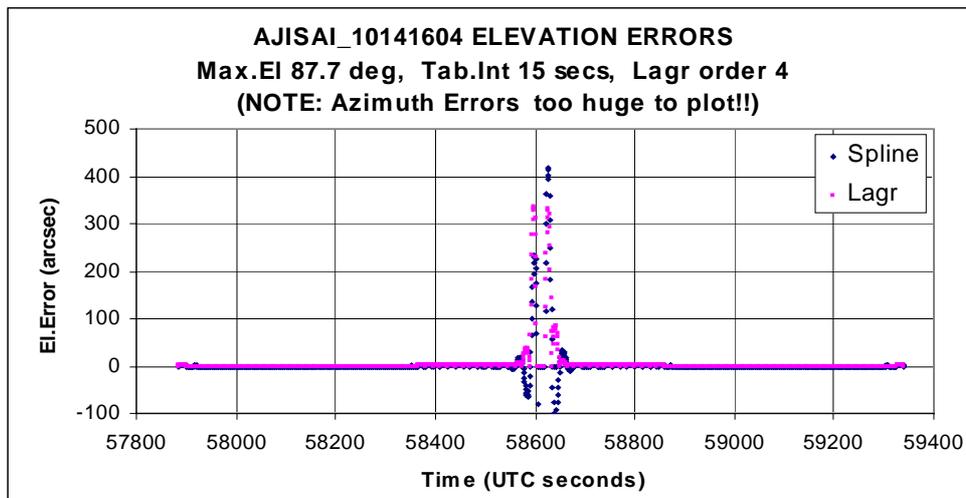


Figure 7: Elevation errors from interpolation errors near TCA on a high AJISAI pass.

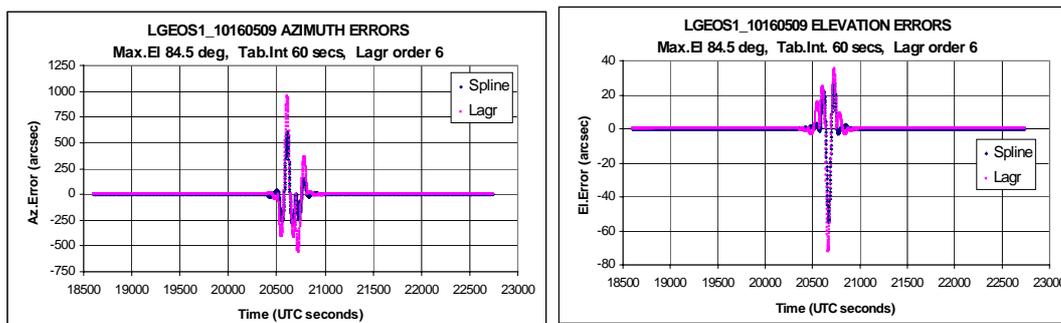


Figure 8: Interpolation-induced errors on a high LAGEOS pass.
(a) Azimuth. (b) Elevation.

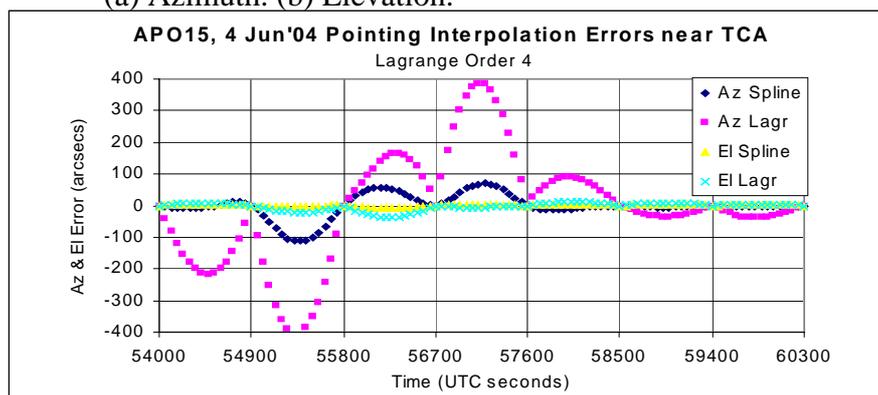


Figure 9: The worst of interpolation errors in azimuth and elevation angles during an Apollo-15 "pass", selected around time of maximum elevation.

ADDENDUM 2: GENERAL NOTE ON INTERPOLATION DEGREE

In Step (3) of the NP Algorithm description, it is noted: “. . while not introducing spurious high-frequency signals (as can occur by fitting a high-order polynomial).” As part of a wider study on interpolation techniques and limitations, I fitted polynomials of degree up to 19 to the 1-second predictions (NOT the Prediction Residuals!) in azimuth, elevation and range, and in topocentric X,Y,Z (East, North, Up), over entire passes using Least Squares. The results are summarized in Table 2 in terms of the RMS of the (Raw - Fitted) differences. Consistent with the remarks in SOLUTIONS 4. above, it was found that such fits converged in X,Y,Z at degrees 8 to 12, while azimuth, elevation and range did not converge even up to degree 19, although the (truncated) ETALON pass came close and actually converged in range at degree 9.

TABLE 2: Least Squares Polynomial Fits to Pass Predictions.

PASS	El @ TCA (deg)	Degree at which RMS reaches 5 ps			RMS of Degree 19 Fit		
		X (E)	Y (N)	Z (U)	Az (arcsec)	El (arcsec)	Range (nanosec)
AJISAI-A	87.7	10	9	10	32589	3373	374.4
AJISAI-B	72.8	12	12	11	3573	865	299.0
LGEOS1-A	84.5	10	10	11	12165	1255	922.3
LGEOS1-B	53.7	10	10	10	19	7	119.6
LGEOS1-C	31.0	10	8	10	6	1	136.6
ETALON1	62.8	6	6	7	(0 at degr 6)	(0 at degr 6)	(0.034 at degr 9)
STELLA-A	37.8	9	9	9	44	30	238.5
STELLA-B	73.5	9	10	10	6598	1835	> 1000
CHAMP-A	25.4	8	8	8	16	13	105.6
CHAMP-B	51.7	8	8	8	1386	776	> 1000

There is a clear correlation between badness-of-fit and maximum elevation when fitting to azimuth and elevation.

To illustrate the spurious signals appearing after convergence, Fig. 10 shows plots of (Raw - Fitted) differences in X for pass STELLA-A in which the RMSs were 0 ps for degrees 9-12. It is interesting to note that the divergence comes from the beginning and end of the pass rather than from periodicities within.

As a side comment, the condition number (κ) of the Normal Matrix inversion process increases vastly with increasing fit degree, and is unacceptably large at $d = 2$ anyway. This indicates that the coefficients of any of the fitted polynomials are absurdly highly correlated. Fitting Chebyshev polynomials gave condition numbers $\kappa \approx 1.0$ right up to degree 19, as expected from the theory of orthogonal polynomials, which is the ideal and indicates that the coefficients are completely uncorrelated. However, they did not change the RMSs one little bit.

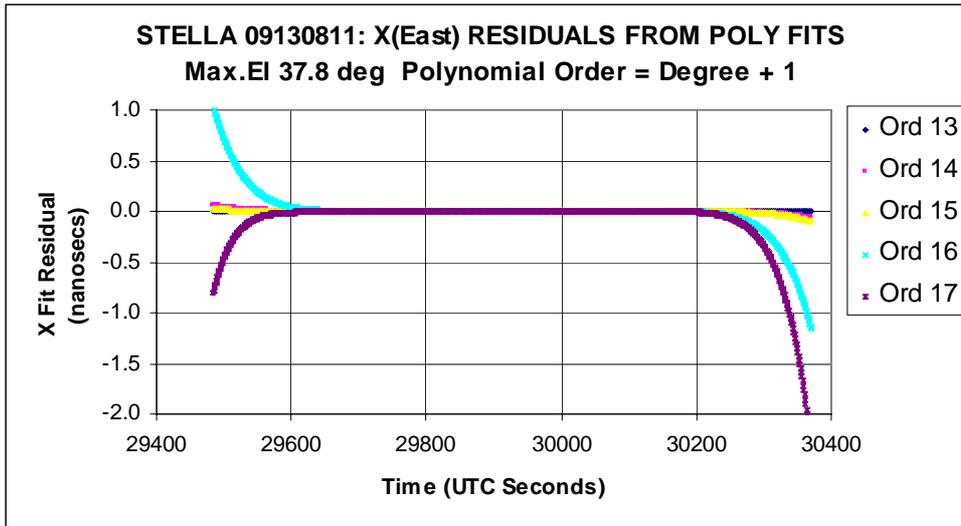


Figure 10: Residuals of predictions in X (Raw - Fitted) for increasing orders of fit after convergence which occurs at degree 9 (order 10).

ADDENDUM 3: MAXIMUM TABULAR INTERVALS

The proposed ILRS Consolidated Predictions Format (CPF) (*Ricklefs, 2004*) advocates the dissemination of predictions in body-fixed X,Y,Z coordinates, at nominated tabular intervals per satellite, to replace IRVs. The magnitudes of interpolation errors as functions of tabular interval and of order were studied, ignoring all other error sources, in 4 coordinate systems:

I: Celestial True-of-Date X,Y,Z, using data at 1-second intervals, again kindly supplied by Chris Moore. These constituted (quasi-) inertial coordinates.

G: Body-fixed in Greenwich X,Y,Z, emulating the CPF scheme. They were generated from “I” by rotating through sidereal time. UT1-UTC and polar motion were ignored.

T: Topocentric East-North-Up. They were generated from “G” by the usual transformations involving geodetic latitude, longitude and ellipsoid height for the new Stromlo station 7825.

P: Polar, i.e. range, azimuth and elevation at the station, generated from “T” by equations (3).

The interpolation errors were characterized by what I loosely call their RSS values (actually, RMS about zero mean), for this purpose defined by:

$$RSS(range) = \sqrt{\sum_{i=1}^n (\text{Interpolated range} - \text{True range})^2 / n}$$

where the sum is over all 1-second points in about a day; and similarly for azimuth and elevation. The results for LAGEOS are shown in Fig. 11 as log-log graphs against tabular intervals of 15, 30, 60, 120, 240, 480 and 960 seconds using the equal-spaced Lagrange formulae for orders 4, 6, 8, 10 and 12. Similar graphs were obtained for CHAMP, STARLETTE, AJISAI and GPS35. The “floors” in the graphs are due to rounding errors when subtracting large numbers, and are very negligible. Otherwise, the curves for Cartesian frames are linear, consistent with equation (2). For any given order, the graphs when interpolating into the three Cartesian systems I, G and T are indistinguishable, and are all far better than for the Polar system P.

Table 3 gives the tabular intervals read from the graphs required to keep the interpolation errors (RSS) below 1 ns (just for target acquisition) and 10 ps (for NP generation) in range, and 1 second of arc in azimuth and elevation. It is evident that the CPF Recommendations are satisfactory, provided that order 8 is used on the body-fixed Cartesian predictions to be supplied.

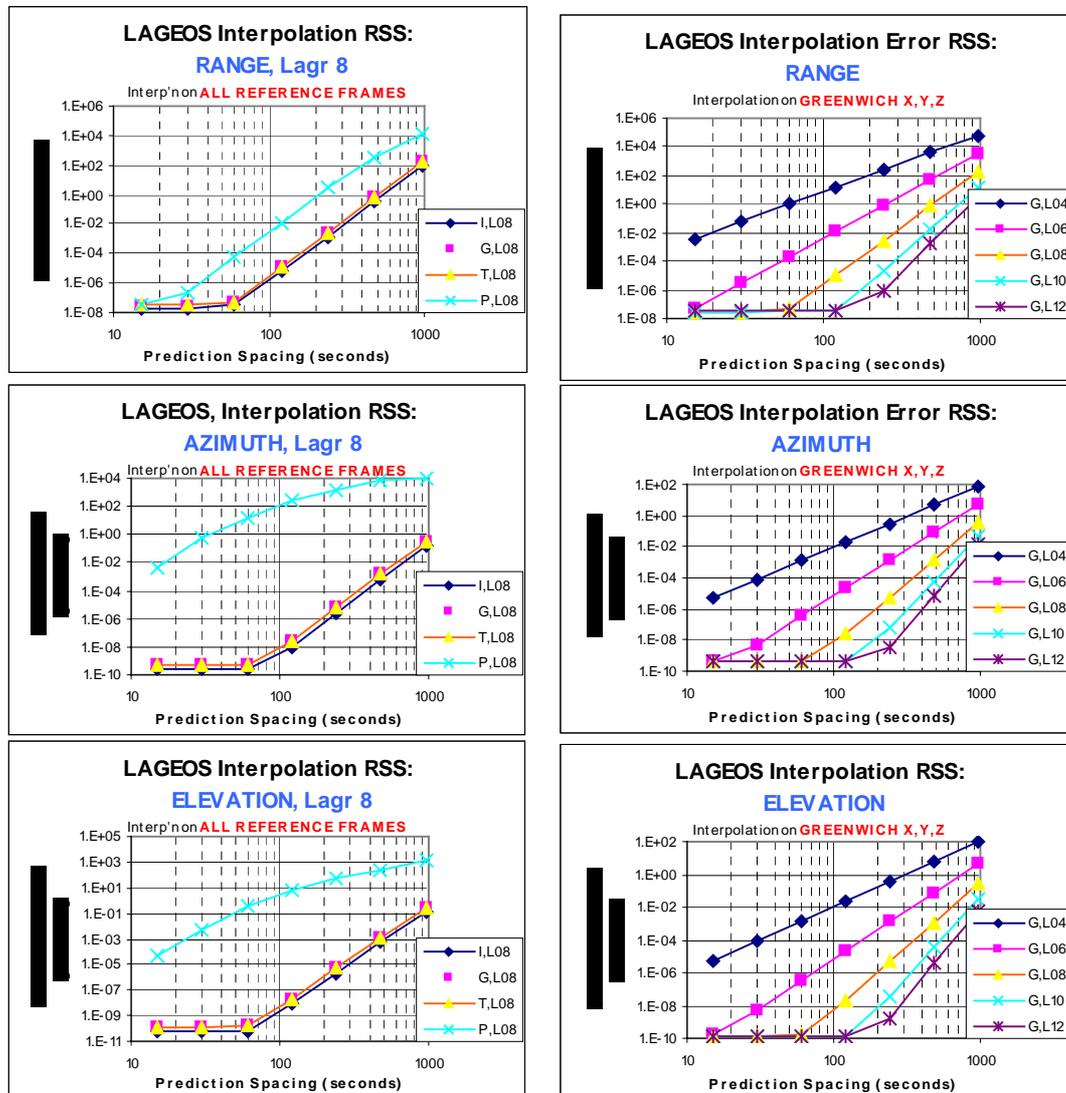


Figure 11: Typical errors due only to interpolation, for LAGEOS over approx. 1 day.

Table 3: Tabular intervals to yield required error specifications

Satellite	Maximum Grid Spacings (seconds) when using 8 th -order Lagrange Interpolation				CPF Rec.	
	RANGE		AZIMUTH	ELEVATION	Deg 7	Deg 9
	1 ns	10 ps	1 arcsec	1 arcsec		
Champ	234	127	441	456	120	180
Starlette	240	127	466	519	180	240
Ajisai	310	170	617	628	240	300
Lageos	501	280	1097	1118	300	600
GPS35	1360	763	2970	3160	900	1800

ADDENDUM 4: NEWTON'S FORWARD DIFFERENCE FORMULA

As is well known, the Lagrange formulation reduces to (*inter alia*) Newton's Forward Difference Formula when the tabular points are equally spaced. Let $f_i \equiv f(t_i)$ be the given values at the tabular points surrounding the interpolation point t , let h be the tabular interval, let $\Delta^n f_k$ be the n^{th} forward difference at the first tabular point t_k used, and let $x = (t - t_k) / h$, so $0 \leq x < k - 1$. Then the interpolated value $y(t)$ is given by:

$$y(t) = f_k + x \Delta f_k + \frac{x(x-1)}{2!} \Delta^2 f_k + \frac{x(x-1)(x-2)}{3!} \Delta^3 f_k + \cdots + \frac{x(x-1) \cdots (x-n+2)}{(n-1)!} \Delta^{n-1} f_k$$

t_k should be chosen so that t falls in the middle interval used, so for example x will lie between 2.0 and 3.0 for a 6-point formula ($n = 6$). The beauty of this is that a matrix of coefficients can be pre-computed to enable rapid calculation in real-time by Horner's Rule. Thus, at each tabular point k pre-compute $c_{k,i}$ as:

$$c_{k,i} = \Delta^{i-1} f_k / (i-1)!, \quad i = 1, \dots, n, \quad k = 1, \dots, \text{num.tab.pts}$$

Then in real-time at time t choose the best k , calculate x and obtain $y(t)$:

$$y(t) = c_{k,1} + x[c_{k,2} + (x-1)[c_{k,3} + \cdots + (x-n+2)c_{k,n}]\cdots].$$

This pre-computation/Horner rapid-evaluation technique has been claimed as an advantage for cubic splines, but it can be applied to any equally-spaced method.